Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis -I

Back paper Examination | Date : 26 December 2022

Maximum marks: 100 | Time: 3 hours

Instructor: B V Rajarama Bhat

Notation: In the following when intervals $[a, b]$, (a, b) or $[a', b']$ are considered it is assumed that $a, b, a', b' \in \mathbb{R}$ and $a < b, a' < b'$.

(1) (i) Show that the set of continuous functions h from $[0, 1]$ to $\mathbb{R}$ such that $h(\frac{1}{2}) = 0$ is uncountable.
(ii) Show that the set of continuous functions h on $[0, 1]$ such that $h(t)$ is an integer for every t in $[0, 1]$ is countable. [15]

(2) State and prove Bolzano-Weierstrass theorem for sequences of real numbers. [15]

(3) Compute liminf and limsup for following sequences.
(i) $a_n = -2 + \frac{n^2-5}{3n^3+2n+4}$ for $n \geq 1$.
(ii) $b_n = 5 + (-\frac{1}{2})^n + (-1)^n 3$ for $n \geq 1$.
(iii) $c_n = \frac{4}{3+(-\frac{2}{3})^n}$ for $n \geq 1$.
[15]

(4) Let $v : \mathbb{R} \to \mathbb{R}$ be a function satisfying $v(3x) = v(x)$ for all $x \in \mathbb{R}$. (i) If v is continuous show that it is a constant function. (ii) Give an example to show that this may not be true if v is discontinuous. [15]

(5) Let $h : (a, b) \to \mathbb{R}$ be a uniformly continuous function.
(i) Show that if $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (a, b), $\{h(x_n)\}_{n\geq 1}$ is Cauchy.
(ii) Show that $\lim_{x\to a+} h(x)$ exists. [15]

(6) Let $n \in \mathbb{N}$. Let $f : [a, b] \to \mathbb{R}$ be a function n-times differentiable on $[a, b]$ and $f^{(n)}(t) = 0$ for every $t \in [a, b]$. Show that f is a polynomial. [15]

(7) Let $f : [a, b] \to \mathbb{R}$ and $g : [a', b'] \to \mathbb{R}$ be functions with $f([a, b]) \subseteq [a', b']$. Take $h = g \circ f$. Suppose $c \in [a, b]$ and $c' = f(c)$. Assume that f is differentiable at c and g is differentiable at c'. Show that h is differentiable at c and

$$h'(c) = g'(c')f'(c).$$

[15]